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Ising-like critical phenomena in two-dimensional percolation: numerical evidence

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Abstract. A recent analytic theory of two-dimensional isotropic percolation indicates that the singular behaviour near p_c is identical to that appropriate for the associated dilute Ising critical point. A specific prediction, supported by recent series expansion studies, is that the mean number of clusters, $K(p)$, presents a singularity of the form $K(p) \sim |p-p_c|^2 \ln|\ln|p-p_c||$, rather than the accepted form $K(p) \sim |p-p_c|^{2-\alpha}$ with $\alpha = -\frac{2}{3}$. A novel numerical and renormalisation group finite-size scaling analysis of the nature of the singularity in $K(p)$ is presented in support of the new theory, which implies the absence of a separate universality class for two-dimensional percolation processes. This study is consistent with the effective values of the exponents $\alpha = 0$ and $\nu = 1$.

1. Introduction

In a recent paper (Jug 1986) I have presented a novel series expansion analysis for the critical singularity of the mean number of clusters per site, $K(p)$, of two-dimensional (2D) percolation. This analysis provides support for a recent theoretical prediction (Jug 1984) according to which the leading singularity in $K(p)$ near the percolation threshold p_c is of the form

$$K(p) \sim |p-p_c|^2 \ln|\ln|p-p_c||. \tag{1.1}$$

This form is to be contrasted with the accepted singular behaviour (see, e.g., Essam 1980)

$$K(p) \sim |p-p_c|^{2-\alpha} \tag{1.2}$$

with $\alpha = -\frac{2}{3}$. This 'conventional' value of α follows from the use of the hyperscaling relation $2-\alpha = d\nu$ and of the conjectured 'exact' value of the pair connectedness length exponent $\nu = \frac{4}{3}$ (den Nijs 1979). Direct evidence in support of this value of α is provided by the series expansion analysis of Domb and Pearce (1976). However, I have shown (Jug 1986) that a conventional series analysis which assumes power-law singular behaviour can be misguided by the presence of logarithmic corrections and can yield fictitious effective exponents. Only when a method of series analysis specifically designed to detect logarithmic corrections is used can the true effective value $\alpha = 0$ be extracted from the series expansion for $K(p)$.

In the present paper, I will present an alternative test of the prediction in equation (1.1) by making use of a novel numerical and finite-size scaling analysis of the nature of the singularity in $K(p)$. The general idea of the method is as follows. For an infinite system, a divergence at p_c can be seen by taking at least three derivatives of

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$K(p)$ with respect to p. For a finite linear lattice size L, the quantity $K'''(L, p_c)$ will be finite but will diverge as $L \to \infty$ in a way directly related to the asymptotic behaviour of $K'''(\infty, p)$ as $p - p_c \to 0$. Indeed, if equation (1.2) holds, one would expect a divergence of the form

$$K'''(L, p_c) \sim L^{(1+\alpha)/\nu}$$

for $L \gg 1$. Therefore, a numerical evaluation of $K'''(L, p_c)$ for progressively larger lattice sizes L should give information on the value of the exponents α and ν. The new theory of 2D percolation (which, since it is based on Grassmann path integral techniques (Jug 1984), will be termed GPI theory) does not yet give a specific prediction for ν and the asymptotic behaviour of the connectedness length. However, the general expectation is that the remaining singularities of 2D percolation will have pure Ising exponents and superimposed logarithmic corrections. Thus, hyperscaling should hold and one should have, in the GPI theory, the effective exponents $\alpha = 0$ and $\nu = 1$. This means that in the case of equation (1.1) one expects $K'''(L, p_c) \sim L$ for $L \gg 1$, whereas in the case of equation (1.2) one should have $K'''(L, p_c) \sim L^{1/4}$. Owing to the considerable difference in the power-law behaviour, it should be possible to decide upon one or other of the competing theories at the cost of a relatively modest amount of computational effort.

The remainder of this paper is organised as follows. In § 2 the prediction, equation (1.1), of the GPI theory is derived in some detail. In § 3 the method employed to evaluate $K'''(L, p_c)$ numerically is discussed and the results obtained for the bond percolation problem on the simple quadratic lattice for $8 \leq L \leq 30$ are presented in § 4. The renormalisation group (RG) based finite-size scaling interpretation of these data is also developed in § 4. Section 5 contains a discussion of the results and my conclusions.

2. Grassmann path integral prediction for the singularity of $K(p)$

As is well known (see, e.g., Essam 1980), the geometric properties of percolation can be derived from the zero temperature statistical mechanics of the Ising model on the diluted lattice associated with the percolation problem. In particular, the mean number of clusters per site corresponds to the $T = 0$ entropy per site, $s(p, T)$, of the dilute Ising model through

$$K(p) = s(p, 0)/(k_B \ln 2)$$

with k_B the Boltzmann constant. Using this relationship and a GPI representation for the free energy of the 2D dilute Ising model, I have been able to show (Jug 1984) that the singular part of $K(p)$ is given by

$$K_s(p) \simeq \lim_{n \to 0} n^{-1} \ln \int D\psi \exp S[\psi] \tag{2.1}$$

where the effective action, in terms of the $n \to 0$ Grassmann fields ψ^a, is

$$S[\psi] = \int d^2x \left[\tfrac{1}{2} i \sum_{a=1}^{n} \bar{\psi}^a (m_0 + \not{\partial}) \psi^a - g_0 \left(\sum_a \bar{\psi}^a \psi^a \right)^2 \right]. \tag{2.2}$$

Here, $m_0 \propto p - p_c^{(0)}$, $g_0 = (2+\sqrt{2})/8$ and $\not{\partial} = \gamma_\mu \partial_\mu$, with γ_1 and γ_2 the 2×2 Pauli matrices. It should be pointed out that in equations (2.1) and (2.2) the limit $T = 0$ is taken before

the $n \to 0$ replica limit; however, it can be shown that the same result for the singular form of $K(p)$ is achieved if the limits are taken in the reverse order.

In order to extract the form of the singularity in equation (2.1) it is convenient to introduce an auxiliary field ϕ such that

$$K_s(p) \simeq \lim_{n\to 0} n^{-1} \ln \int \mathrm{D}\psi\, \mathrm{D}\phi \exp S[\psi, \phi]$$

with

$$S[\psi, \phi] = \int \mathrm{d}^2x \left(\tfrac{1}{2}\mathrm{i} \sum_a \bar{\psi}^a (m_0 + \not{\partial}) \psi^a - \tfrac{1}{2}\phi^2 + \mathrm{i} g_0^{1/2} \phi \sum_a \bar{\psi}^a \psi^a \right).$$

One then looks at the behaviour of the equivalent of the specific heat (the sum over replicas and $n = 0$ limit are understood):

$$\begin{aligned} K_s''(p) &\simeq n^{-1} \left[\left\langle \left(\int \mathrm{d}^2x\, \bar{\psi}^a \psi^a \right)^2 \right\rangle - \left\langle \int \mathrm{d}^2x\, \bar{\psi}^a \psi^a \right\rangle^2 \right] \\ &= -n^{-1} \Gamma^{(0,0;2)}(m_0, \lambda_0; k = 0) \end{aligned} \tag{2.3}$$

where $\lambda_0 = g_0^{1/2}$ and where $\Gamma^{(N,M;L)}$ is the vertex part of the Green function of N ψ fields, M ϕ fields and L mass insertions $\int \mathrm{d}^2x\, \bar{\psi}^a \psi^a \exp(\mathrm{i}kx)$. Renormalised perturbation theory and the renormalisation group (Brézin *et al* 1976, Amit 1978) are used at this point in order to extract the singular behaviour of equation (2.3). The renormalised vertex function $\Gamma^{(0,0;2)}$ satisfies the equation

$$\left(\kappa \frac{\partial}{\partial \kappa} + \beta(\lambda) \frac{\partial}{\partial \lambda} - \theta(\lambda) m \frac{\partial}{\partial m} - 2\theta(\lambda) \right) \Gamma^{(0,0;2)}(m, \lambda, \kappa; k) = B(\lambda) \tag{2.4}$$

where κ^{-1} is the (arbitrary) renormalisation length scale and $B(\lambda)$ a smooth function of λ only. $m \propto p - p_c$ and λ are the renormalised mass and (dimensionless) coupling constant, respectively, and the renormalisation functions $\beta(\lambda)$ and $\theta(\lambda)$ have the small coupling expansion

$$\begin{aligned} \beta(\lambda) &= \tfrac{1}{2} a \lambda^3 + \mathrm{O}(\lambda^4) \\ \theta(\lambda) &= b\lambda^2 + \mathrm{O}(\lambda^3) \end{aligned} \tag{2.5}$$

with a and b n-dependent constants. For $n = 0$, one has the relationship

$$b/a = -\tfrac{1}{2} \tag{2.6}$$

which follows from the renormalised version of the equation of motion $\partial S[\psi, \phi]/\delta\phi(x) = 0$ and the fact that for $n = 0$ the field ϕ has zero anomalous dimension. The standard solution of equation (2.4) (Brézin *et al* 1976) is

$$\begin{aligned} &\Gamma^{(0,0;2)}(m, \lambda, \kappa; 0) \\ &= \exp\left(-\int_1^\rho \frac{\mathrm{d}x}{x} 2\theta(\lambda(x)) \right) \Gamma^{(0,0;2)}(m(\rho)\rho^{-1}, \lambda(\rho), \kappa; 0) \\ &\quad - \int_1^\rho \frac{\mathrm{d}x}{x} B(\lambda(x)) \exp\left(-\int_1^x \frac{\mathrm{d}y}{y} 2\theta(\lambda(y)) \right) \end{aligned} \tag{2.7}$$

where $m(\rho)$ and $\lambda(\rho)$ are the running mass and coupling constant, given by

$$\rho = \exp\left(\int_\lambda^{\lambda(\rho)} \mathrm{d}\lambda'/\beta(\lambda') \right) \tag{2.8}$$

and by

$$m(\rho) = m \exp\left(-\int_1^{\rho} \frac{dx}{x} \theta(\lambda(x)) \right) \tag{2.9}$$

respectively. The marginal RG fixed point, $\lambda^* = 0$, is approached logarithmically as $\rho \to 0$; equation (2.8), together with equation (2.5), gives the asymptotic behaviour

$$\lambda(\rho) \simeq (-a \ln \rho)^{-1/2} \to \lambda^* = 0. \tag{2.10}$$

At the same time, equation (2.9), together with equations (2.5) and (2.10), yields, for $\rho \to 0$,

$$m(\rho) \simeq m(-\ln \rho)^{b/a} = m(-\ln \rho)^{-1/2} \tag{2.11}$$

where use has been made of equation (2.6). Also in the same limit, one has

$$\begin{aligned} &\exp\left(-\int_1^{\rho} \frac{dx}{x} 2\theta(\lambda(x)) \right) \simeq (-\ln \rho)^{-1} \\ &\int_1^{\rho} \frac{dx}{x} B(\lambda(x)) \exp\left(-\int_1^{x} \frac{dy}{y} 2\theta(y) \right) \simeq -B(0) \ln(-\ln \rho). \end{aligned} \tag{2.12}$$

The small but otherwise arbitrary scaling parameter ρ is now fixed, for an infinite system, by the condition

$$m(\rho^*)(\kappa\rho^*)^{-1} = 1 \tag{2.13}$$

that is, by virtue of equation (2.11):

$$\rho^* \simeq m\kappa^{-1} |\ln|m\kappa^{-1}||^{-1/2} \tag{2.14}$$

which vanishes as $m \propto p - p_c \to 0$. Therefore, using equations (2.12)-(2.14), equation (2.7) yields the asymptotic behaviour

$$\begin{aligned} \Gamma^{(0,0;2)}(m, \lambda, \kappa; 0) &\simeq (-\ln \rho^*)^{-1} \Gamma^{(0,0;2)}(\kappa, 0, \kappa; 0) + B(0) \ln(-\ln \rho^*) \\ &\simeq B(0) \ln|\ln|m|| \{1 + O[(\ln|m|\ln|\ln|m||)^{-1}]\}. \end{aligned} \tag{2.15}$$

Bearing in mind that $B(0)/n$ is finite for $n = 0$, equations (2.3) and (2.15) give the leading singular behaviour

$$K_s''(p) \simeq \ln|\ln|p - p_c|| \tag{2.16}$$

which in turn implies the result of equation (1.1) for $K(p)$. Equation (2.16) represents an extension to the percolation threshold of an earlier theory proposed for the critical behaviour of the 2D dilute Ising model in the low dilution limit (Dotsenko and Dotsenko 1983).

To conclude this section, I will remark that the validity of equation (2.16), and of the GPI approach to 2D percolation in general, depends on the validity of the perturbation expansion in g_0 and of the corresponding renormalised expansion in g. It is always possible that this perturbation theory breaks down precisely in the $T = 0$, $p = p_c$ limits of the RG calculation for the dilute Ising model. However, no sign of such a breakdown has emerged thus far and no direct way of determining the range of validity of such a perturbation theory is yet available. This validity is an assumption common to all perturbative RG calculations and it is effectively this assumption that is being tested in the present (and previous (Jug 1986)) paper.

3. Monte Carlo evaluation of $K'''(L, p_c)$

As outlined in the introduction, the nature of the singularity in $K(p)$ can be analysed numerically by looking at $d^3K(L, p)/dp^3$ at $p = p_c$ and as a function of the lattice size L. Looking at this third derivative has the advantages of generating a divergence as $p \to p_c$ and $L \to \infty$ and of eliminating the analytic background in $K(p)$, so that K''' has no regular part that requires subtraction.

In order to evaluate derivatives of $K(p)$ it is most convenient to make use of fluctuation formulae, expressing derivatives in terms of cumulants of appropriate correlation functions. This is done, in the case of, say, a bond percolation problem, by introducing bond occupation variables $[b_l]$, with $b_l = 0$ for an unoccupied bond l and $b_l = 1$ for an occupied bond, and by writing

$$K(L, p) = \langle n_c \rangle = \sum_{[b_l]} n_c([b_l])\, W([b_l]) \Big(\sum_{[b_l]} W([b_l]) \Big)^{-1}. \tag{3.1}$$

Here, $n_c([b_l])$ is the number of clusters per site for a given configuration $[b_l]$ of bonds and the associated statistical weight is given by

$$W([b_l]) = \exp\Big(g(p) \sum_l b_l \Big) \tag{3.2}$$

where $g(p) = \ln(p/(1-p))$. Direct differentiation of equations (3.1) and (3.2) then gives

$$\begin{aligned} K'''(L, p) = {} & g'''(p)(\langle n_c N_l \rangle - \langle n_c \rangle \langle N_l \rangle) \\ & + 3g''(p)g'(p)(\langle n_c N_l^2 \rangle - 2\langle n_c N_l \rangle \langle N_l \rangle + 2\langle n_c \rangle \langle N_l \rangle^2 - \langle n_c \rangle \langle N_l^2 \rangle) \\ & + [g'(p)]^3(\langle n_c N_l^3 \rangle - 3\langle n_c N_l^2 \rangle \langle N_l \rangle + 6\langle n_c N_l \rangle \langle N_l \rangle^2 - 6\langle n_c \rangle \langle N_l \rangle^3 \\ & + 6\langle n_c \rangle \langle N_l^2 \rangle \langle N_l \rangle - 3\langle n_c N_l \rangle \langle N_l^2 \rangle - \langle n_c \rangle \langle N_l^3 \rangle) \end{aligned} \tag{3.3}$$

where $N_l = \Sigma_l b_l$ is the total number of occupied bonds. For a site percolation problem, the total number of occupied sites $N_s = \Sigma_s b_s$ would replace N_l in equation (3.3). For the special case in which $p_c = \frac{1}{2}$, such as the bond percolation problem on the simple quadratic lattice considered in this paper, equation (3.3) becomes

$$\begin{aligned} K'''(L, p_c) = {} & 32(\langle n_c N_l \rangle - \langle n_c \rangle \langle N_l \rangle) + 64(\langle n_c N_l^3 \rangle - 3\langle n_c N_l^2 \rangle \langle N_l \rangle + 6\langle n_c N_l \rangle \langle N_l \rangle^2 - 6\langle n_c \rangle \langle N_l \rangle^3 \\ & + 6\langle n_c \rangle \langle N_l^2 \rangle \langle N_l \rangle - 3\langle n_c N_l \rangle \langle N_l^2 \rangle - \langle n_c \rangle \langle N_l^3 \rangle). \end{aligned} \tag{3.4}$$

Thus, using equation (3.4) it is possible to determine $K'''(L, p_c)$ by generating random configurations of bonds with average occupation probability $p = p_c$ on a computer and by evaluating n_c and N_l for each configuration. Owing to the large statistical noise associated with the third-order cumulant in equation (3.4), a considerable number ($\geqslant 2 \times 10^7$) of independent configurations has to be generated and analysed in order to evaluate $K'''(L, p_c)$. The averages over all possible bond configurations required in equation (3.4) are approximated by averaging over a sample of up to 10^6 configurations. The measurement of $K'''(L, p_c)$ is then repeated at least 20 times, so that a statistics can be constructed, making sure that the resulting average value and the value obtained after averaging over the total number of configurations are indistinguishable. The numerical data reported in this paper exploit an algorithm for the evaluation of n_c which assigns the occupied bonds and searches for the connected clusters by scanning the lattice sequentially site by site. This results in a relatively slow counting of the

clusters and, consequently, in drastic limitations on the maximum available lattice size compatible with reasonable statistics. The maximum size available in this paper is $L = 30$ and this lattice alone required more than 300 h CPU time on a Cray-1 computer. It is possible that by making use of parallel processing the counting of clusters may be made through a few operations on the whole of the lattice, thus reducing considerably the CPU time per configuration and making possible the extension of the present analysis to much larger lattices.

Four sets of data are presented in this paper. They all refer to bond percolation clusters on the simple quadratic $L \times L$ lattice with periodic boundary conditions. Triangles refer to a $K(p)$ which includes linked clusters only; circles refer to $K(p)$ for all clusters, including unlinked sites. (○) and (△) refer to the true $K'''(L, p_c)$, for which the statistical noise becomes rapidly uncontrollable with increasing L. (●) and (▲) refer to a modified counting algorithm which was found to be considerably less noisy. In the modified algorithm, whenever clusters n and $n-1$ merge into a single cluster in the sequential filling of the lattice bonds, cluster $n-2$ is neglected in the counting. It will be assumed that the neglected clusters are not spanning clusters and that therefore the modified $\tilde{K}(p)$ has the same singular behaviour as the original $K(p)$, i.e. $\tilde{K}'''(L, p_c) \propto K'''(L, p_c)$. This is strikingly confirmed by the fact that all sets of data settle down to the same asymptotic behaviour for $L \gg 1$, as shown in figures 1, 2 and 3.

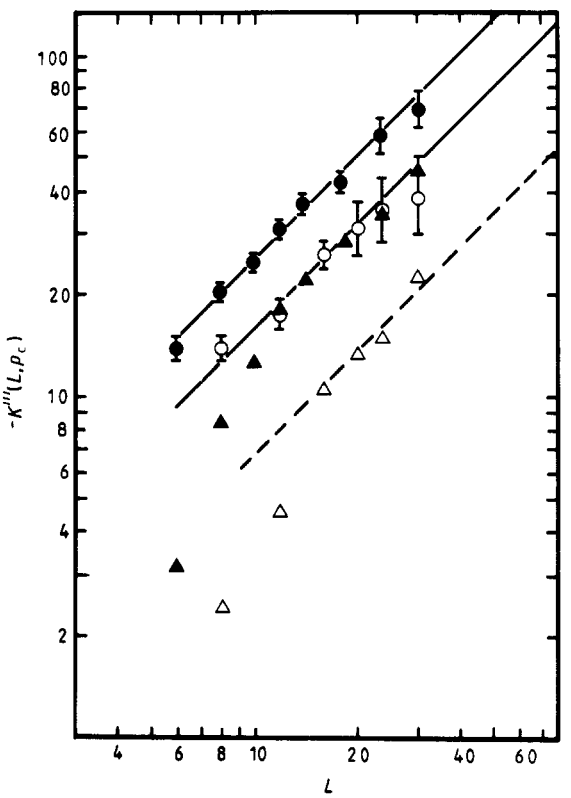

Figure 1. Double logarithmic plot of the divergence of $K'''(L, p_c)$ with lattice size L. See text for explanation of symbols. Straight lines have slope = 1 and are guides to the eye for the presumed asymptotic behaviour for $L \gg 1$.

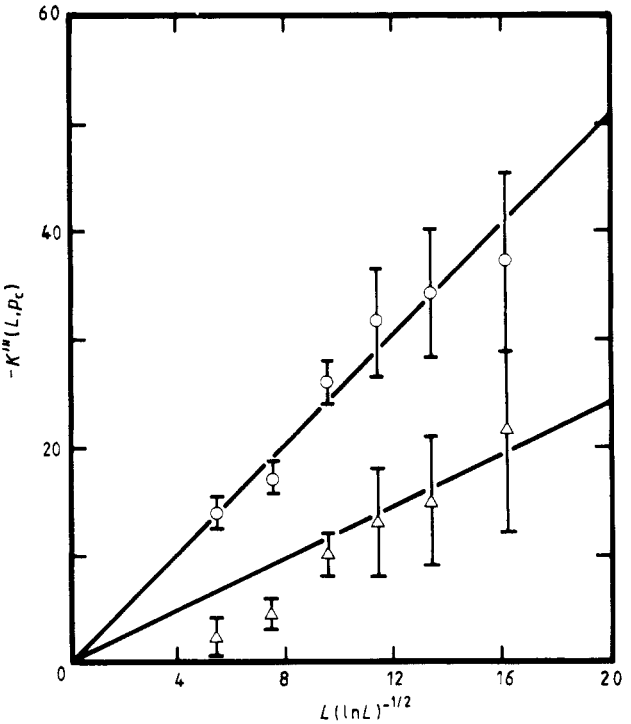

Figure 2. Plot of $K'''(L, p_c)$ against $L(\ln L)^{-1/2}$ to verify agreement with the full GPI prediction, equation (4.14). The pattern of convergence is identical to that of figure 1.

In figure 4 I present the histogram relative to my evaluation of $K'''(L, p_c)$ for $L = 16$ (all clusters counted), with the continuous curve representing the probability distribution function as determined from maximum entropy methods (Collins and Wragg 1977). The resulting distribution is Gaussian and fits the histogram well, with mean and standard error as reported for the corresponding data point in figures 1 and 2. The statistics relative to any of the remaining data points is of comparable quality.

4. Finite-size scaling interpretation

One assumes, to start with, that the critical behaviour near p_c is determined by an ordinary RG fixed point, as is commonly believed. Then, with $\alpha = -\frac{2}{3}$, the first divergent derivative of $K(p)$ is

$$K'''(p) \simeq |p - p_c|^{-1-\alpha} \tag{4.1}$$

with K''' a multiplicatively renormalisable quantity in any of the available field-theoretic formulations of percolation. The RG-based finite-size scaling theory (Brézin 1982) allows one to write, for a system of size $L \gg 1$,

$$K'''(L, p) = K'''(\infty, p)F(L/\xi) = K'''(\infty, p)f((p - p_c)L^{1/\nu}) \tag{4.2}$$

with F and f universal functions and ξ the connectedness length of the infinite system. Then, equations (4.1) and (4.2) imply the asymptotic behaviour

$$K'''(L, p_c) = AL^{(1+\alpha)/\nu} = AL^{1/4} \tag{4.3}$$

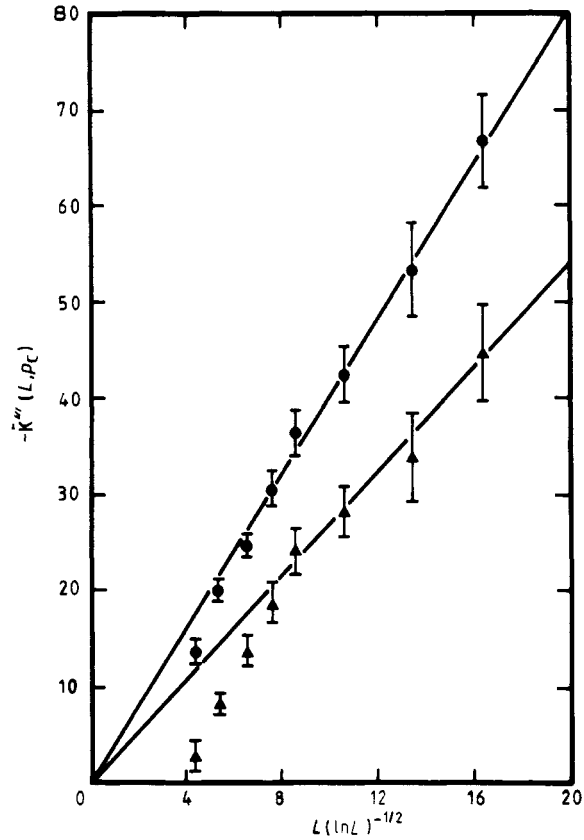

Figure 3. Same as in figure 2, but for the modified $\tilde{K}(L, p)$.

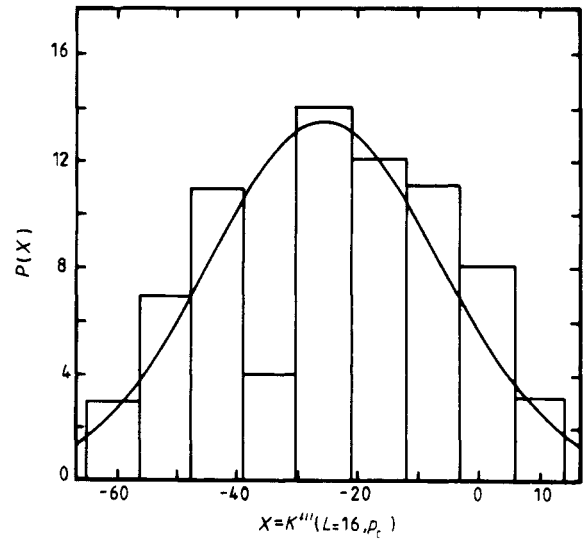

Figure 4. Histogram and continuous probability distribution function $P(X)$ for $X = K'''(L = 16, p_c)$. $P(X)$ is normalised to the total number of independent evaluations, $N = 73$, and each evaluation corresponds to averages over 310 000 configurations.

where A is a non-universal constant and where the conventional values of the exponents α and ν have been used.

If, on the other hand, 2D percolation is dominated by a marginal RG fixed point, as in the GPI theory, one expects (Brézin 1982, Brézin and Zinn-Justin 1985) a breakdown of the ordinary finite-size scaling prediction, equations (4.2) and (4.3), though only through the appearance of logarithmic corrections in the L dependence. The specific predictions of the finite-size GPI theory for $K'''(L, p)$ will now be derived for the first time. It is clear that, in the spirit of the Grassmann field theory of § 2, K''' corresponds to the vertex function

$$K'''(L, p) \simeq n^{-1}\Gamma^{(0,0;3)}(L, m_0, \lambda_0; k_i = 0). \tag{4.4}$$

This function is multiplicatively renormalisable and satisfies the RG equation

$$\left(\kappa\frac{\partial}{\partial\kappa} + \beta(\lambda)\frac{\partial}{\partial\lambda} - \theta(\lambda)m\frac{\partial}{\partial m} - 3\theta(\lambda)\right)\Gamma^{(0,0;3)}(L, m, \lambda, \kappa; k_i) = 0 \tag{4.5}$$

by virtue of the fact that no new renormalisation is associated with the length L of the periodic finite-size lattice (Brézin 1982). The solution of equation (4.5) has the form

$$\Gamma^{(0,0;3)}(L, m, \lambda, \kappa; k_i)$$
$$= \rho^{-1}\exp\left(-\int_1^\rho \frac{dx}{x}3\theta(\lambda(x))\right)\Gamma^{(0,0;3)}(L, m(\rho)\rho^{-1}, \lambda(\rho), \kappa; k_i\rho^{-1}) \tag{4.6}$$

or, using dimensional analysis,

$$\Gamma^{(0,0;3)}(L, m, \lambda, \kappa; k_i)$$
$$= \exp\left(-\int_1^\rho \frac{dx}{x}3\theta(\lambda(x))\right)\Gamma^{(0,0;3)}(L, m(\rho), \lambda(\rho), \kappa\rho; k_i) \tag{4.7}$$

where for small ρ one has, using the results of § 2,

$$\exp\left(-\int_1^\rho \frac{dx}{x}3\theta(\lambda(x))\right) \simeq (-\ln\rho)^{-3/2}. \tag{4.8}$$

For an infinite system, ρ is fixed as in equation (2.13); then, using equation (2.14), equations (4.6) and (4.8) yield

$$\Gamma^{(0,0;3)}(\infty, m, \lambda, \kappa; 0) \simeq \rho^{*-1}(-\ln\rho^*)^{-3/2}\Gamma^{(0,0;3)}(\infty, \kappa, 0, \kappa; 0)$$
$$\simeq Cm^{-1}(\ln|m\kappa^{-1}|)^{-1}.$$

By virtue of equation (4.4) and the fact that C is a constant of $O(n)$, this implies

$$K'''(\infty, p) \simeq |p - p_c|^{-1}(\ln|p - p_c|)^{-1} \tag{4.9}$$

which is consistent with equations (1.1) and (2.16); equation (4.9) is the GPI alternative to equation (4.1). On the other hand, for a finite-size system one may consider the dimensionless ratio

$$\Phi(L, m, \lambda, \kappa) = \Gamma^{(0,0;3)}(L, m, \lambda, \kappa; 0)/\Gamma^{(0,0;3)}(\infty, m, \lambda, \kappa; 0)$$
$$= K'''(L, p)/K'''(\infty, p) \tag{4.10}$$

which, according to equation (4.7), is an invariant of the RG:

$$\Phi(L, m, \lambda, \kappa) = \Phi(L, m(\rho), \lambda(\rho), \kappa\rho) = \tilde{\Phi}(L\kappa\rho, m(\rho)/(\kappa\rho), \lambda(\rho)) \tag{4.11}$$

where, in the last equality, Φ has been written as a function of dimensionless variables. The parameter ρ is fixed according to $\rho^* = (L\kappa)^{-1}$, which implies, from equation (2.11),

$$m(\rho^*) \simeq m[\ln(L\kappa)]^{-1/2}. \tag{4.12}$$

It then follows from equations (4.10)-(4.12) that, for $L \gg 1$,

$$\begin{aligned} K'''(L, p) &= K'''(\infty, p)\tilde{\Phi}(1, mL[\ln(L\kappa)]^{-1/2}, 0) \\ &= K'''(\infty, p) f((p - p_c)L(\ln L)^{-1/2}) \end{aligned} \tag{4.13}$$

which shows the breakdown of simple finite-size scaling and is the GPI version of equation (4.2). Using equations (4.9) and (4.13), one arrives at the following GPI prediction for the asymptotic behaviour of $K'''(L, p_c)$

$$K'''(L, p_c) = AL(\ln L)^{-1/2} \tag{4.14}$$

which, compared with equation (4.3), shows that in the GPI theory $\alpha = 0$ and $\nu = 1$ are the effective 'thermal' exponents of 2D percolation. This implies that the hyperscaling relation $2 - \alpha = d\nu$ is verified also in the GPI approach when the effective values of the exponents are used.

Given the considerable difference between the competing predictions, equations (4.3) and (4.14), for the asymptotic behaviour of $K'''(L, p_c)$, it should be possible to test the GPI theory against the conventional one by using the available numerical data. In figure 1, the double logarithmic plot of $K'''(L, p_c)$ against lattice size L shows, on

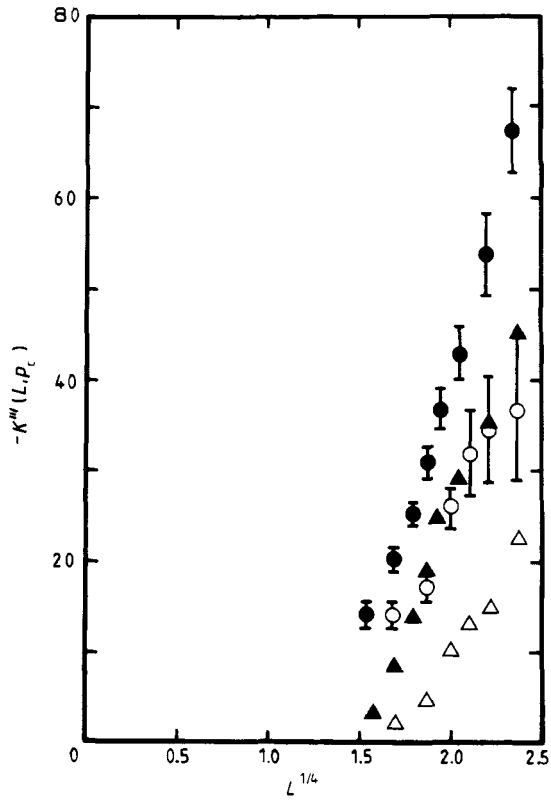

Figure 5. Plot of $K'''(L, p_c)$ against $L^{1/4}$ to test the prediction of the conventional theory, equation (4.3). No straight line passing through the origin can be fitted to the available data.

all sets of data, convergence towards an asymptotic behaviour with effective power-law exponent $(1+\alpha)/\nu = 1$, consistent with the GPI prediction, equation (4.14). In figures 2 and 3, the numerical data for $K'''(L, p_c)$ and for the modified $\tilde{K}'''(L, p_c)$, respectively, are plotted against $L(\ln L)^{-1/2}$ and it can be seen that the same pattern of convergence as in figure 1 towards the full GPI prediction, equation (4.14), is recovered. The available data, plotted against $L^{1/4}$, do not agree with the prediction of the conventional theory, equation (4.3) (see figure 5), although in principle convergence towards this behaviour for much larger values of L cannot be excluded.

It should be noticed at this point that the possible interference in the finite-size analysis of the available data from corrections to the leading scaling behaviour, equation (4.3) or equation (4.14), has not been considered here. This is partly justified by the insufficient quality of the statistics for the present data and by the fact that the present numerical study seeks agreement with one or other of the competing predictions rather than a precise evaluation of a critical exponent. More to the point, in the case of equation (4.14) corrections to scaling terms will be logarithmic in L and are expected to be irrelevant to the analysis.

5. Discussion and conclusions

The available numerical data for $K'''(L, p_c)$ refer to lattices of relatively small size. However, convergence towards the asymptotic behaviour predicted by the GPI theory of 2D percolation is observed quite clearly and this provides alternative evidence that the theory is consistent. It could be argued that recent numerical studies of 2D percolation (Reynolds *et al* 1978, Eschbach *et al* 1981, Blöte *et al* 1981, Derrida and de Seze 1982) have shown that lattices of much larger size than the ones used here must be employed in order to approach the critical behaviour predicted by the conventional theory. However, it is possible that for a free energy property like K''' the convergence to the expected exponents is much faster as a function of increasing L than it is for other percolation properties. Also, the slow convergence observed in these studies (see, e.g., Eschbach *et al* 1981) as a function of increasing lattice or cluster size could be taken as warning of the presence of a marginal RG fixed point (Luck 1984, Brézin and Zinn-Justin 1985). The present study, along with the observation of a marginal operator in a real space RG analysis of the 2D one-state Potts model (Andelman and Berker 1981), suggests that this may just be the case for 2D percolation. It is not impossible that, as in the case of the series expansion analysis (Jug 1986), insisting on an assumed power-law behaviour and an ordinary RG fixed point in the generation and analysis of the numerical data may result in a slow convergence towards a set of apparent critical exponents. This is known to be the case for the 2D four-state Potts model (Barber 1983), where the presence of logarithmic corrections generates fictitious values for the exponents as extracted by conventional numerical methods.

In order to illustrate this possibility one may consider the following. Most of the numerical methods aiming at the determination of the exponent ν exploit the fact that, according to finite-size scaling theory (Fisher 1971), the apparent percolation threshold $\bar{p}_c$ in a lattice of size L approaches the true p_c according to the asymptotic law

$$\Delta p_c = p_c - \bar{p}_c = BL^{-1/\nu}. \tag{5.1}$$

A similar asymptotic behaviour is expected for the L dependence of the finite-size rounding of the transition, δp. If a scaling form such as equation (4.2) or (4.13) is

predicted, then the rounding region should correspond to the condition $x \sim 1$ for the argument of the universal function $f(x)$. Indeed, in the case of the conventional theory, equation (4.2) yields the prediction in equation (5.1) for either Δp_c or δp. Alternatively, equation (4.13) leads to the asymptotic behaviour

$$\delta p \simeq \Delta p_c = BL^{-1}(\ln L)^{1/2} \tag{5.2}$$

for $L \gg 1$. It can be shown that this behaviour agrees with the numerical study by Levinshteĭn *et al* (1976) of site percolation on finite-size square lattices. This study does not employ any real space RG rescaling and is conducted in the same spirit as in the present work. The authors accurately determine the dispersion W_L ($\propto \Delta p_c$ for large L) of percolation thresholds for lattice sizes $1 \leqslant L \leqslant 128$ and find excellent agreement with a finite-size dependence given by $W_L = B(L+C)^{-1/\nu}$, with B and C adjustable parameters and $\nu = 1.33$. On the other hand, it is shown in figures 6(a) and (b) that an equally good fit is obtained by using

$$W_L = BL^{-1}[\ln(CL+1)]^{1/2} \tag{5.3}$$

with $B = 1.18$ and $C = 0.075$. Although much in this argument is heuristic, it indicates that the asymptotic behaviour predicted by the GPI approach, equation (5.2), could be confused with a power law and an exponent $\nu \neq 1$. In a future planned paper, a finite-size GPI calculation of Δp_c will be undertaken. It would be interesting to fit the resulting detailed prediction to numerical data for W_L obtained as in Levinshteĭn *et al* (1976), but for larger lattice sizes.

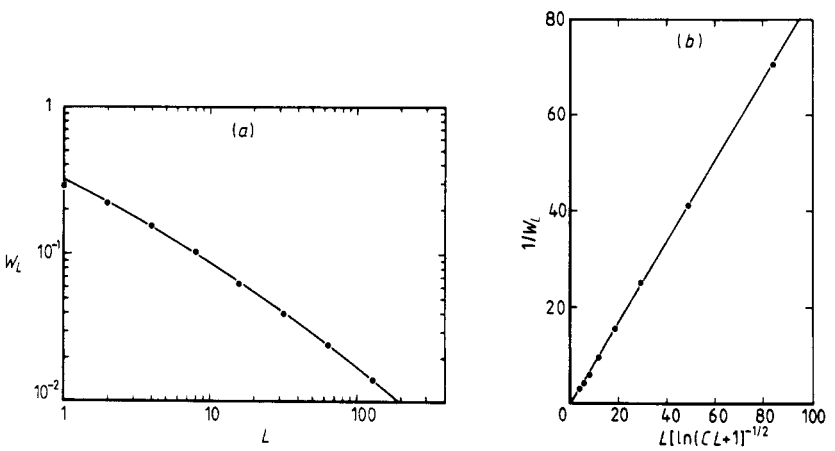

Figure 6. (a) Double logarithmic plot of W_L against L; the curve is a plot of equation (5.3) with $B = 1.18$ and $C = 0.075$. (b) Plot of $1/W_L$ against $L[\ln(CL+1)]^{-1/2}$; the straight line has slope $1/B$, with B and C as in (a). Numerical data are taken from Levinshteĭn *et al* (1976).

Together with the independent confirmation obtained from the series expansion analysis of the singularity in $K(p)$ as given by equation (1.1), the present work indicates quite clearly that the critical properties of 2D percolation may be dominated by a marginal Ising-like RG fixed point. The numerical and finite-size scaling study of 2D percolation presented in this study is the first of its kind and is consequently somewhat preliminary in nature. In order to rule out the conventional behaviour, equation (4.3),

confidently, data on $K'''(L, p)$ for much larger lattice sizes are required and these can be obtained only through the use of a new advanced counting algorithm, adapted for use on an array processor or special purpose computer. More to the point, the predictions of the GPI theory need to be extended to other properties of 2D percolation, e.g. the percolation probability $P(p)$ and the pair connectedness function $C(r, p)$. Various quantities related to these properties lend themselves more easily to analysis through numerical and series expansion methods. Knowledge of the GPI asymptotic behaviour of these properties may lead to a different interpretation of this type of data. Unfortunately, theoretical methods for the analysis of the asymptotic behaviour of $P(p)$ and $C(r, p)$ seem to be unavailable at present. This is due to the fact that in the Grassmann field theory of any Ising model the spin pair correlation function, to which $P(p)$ and $C(r, p)$ are ultimately connected, does not correspond to the pair correlation function $\langle\psi(r)\psi(0)\rangle$ of the Grassmann fields $\psi(x)$, as it would do in any ordinary field theory. Rather, the asymptotic behaviour of the expectation value of a complicated composite operator, e.g.

$$\left\langle \exp\left(\int_0^r \mathrm{d}y\, \bar{\psi}(y)\varphi(y) \right) \right\rangle$$

needs to be extracted for $r \to \infty$ and $m \to 0$. To date, this has not been achieved satisfactorily and uncontroversially, even in the case of the pure 2D Ising model. Thus, progress in the understanding of 2D percolation and many other lattice models may be directly dependent on progress in the GPI formalism for Ising problems.

It is important to remark, in conclusion, that 2D Ising critical exponents have been occasionally observed for 2D percolation in the past, both theoretically (Thorpe and Kirkpatrick 1979) and numerically (Roussenq *et al* 1976). Moreover, Fucito and Parisi (1981) have found evidence for the breakdown of the momentum space RG approach to percolation, which otherwise yields the conventional description, specifically in $d = 2$. These findings strengthen the point of view developed in the present study.

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Note added in proof. H Kesten (1986 *Phys. Rev. Lett.* **56** 1210) has pointed out that the form, equation (1.1), proposed for the singularity in $K(p)$ is in (marginal) disagreement with his rigorous result on the Hölder continuity of this function, namely there exists an $a > 0$ and a $C > 0$ such that, for all $p_1, p_2 \in [0, 1]$,

$$|K''(p_1) - K''(p_2)| \leqslant C|p_1 - p_2|^a.$$

Unless the agreement between the GPI theory and the numerical results for $K(p)$ is a mere coincidence, one may wonder whether the possibility of an $a = 0$ and of a logarithm on the right-hand side of the above inequality has been totally overlooked. It is intriguing that the conditions of $a > 0$ and $\alpha < 0$ should go hand in hand in Kesten's arguments.

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